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m-dimensional lattice sums of generalized hypergeometric functions

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Abstract. Representations and convergence criteria for infinite m-dimensional lattice sums of generalized hypergeometric functions ${}_pF_{p+1}$ are deduced by appealing to the principle of mathematical induction. In particular, we show that such a lattice sum may be expressed essentially as a finite sum of Mellin transforms of products of Bessel functions of order $\frac{1}{2}(m-2)$ and the functions ${}_pF_{p+1}$ in the lattice sum. In addition, a direct derivation for the three-dimensional case is provided. Moreover, we construct a countably infinite class of null-functions on increasingly larger open intervals which are parametrically independent of the functions ${}_pF_{p+1}$ generating them.

1. Introduction

Let $\boldsymbol{q}(m)$ denote the vector whose m components ($m \geqslant 1$) range over all integers (positive, negative and zero) which are usually called Z. The length of the vector $\boldsymbol{q}(m)$, i.e. the square root of the sum of the squares of its components, is denoted by q. We shall consider m-dimensional lattice sums of generalized hypergeometric functions ${}_pF_{p+1}$ ($p \geqslant 0$) defined for $x > 0$ by

$$W(\boldsymbol{\alpha}; x) \equiv \sum_{\boldsymbol{q}(m)} \exp(2\pi \mathrm{i}\boldsymbol{\alpha} \cdot \boldsymbol{q})\, {}_pF_{p+1}[(a_p); (b_{p+1}); -x^2q^2] \tag{1.1a}$$

where the components of the constant vector $\boldsymbol{\alpha}(m)$ are arbitrary real numbers and $\boldsymbol{\alpha} \cdot \boldsymbol{q}$ is the vector dot product. Clearly, since in equation (1.1a), $\boldsymbol{q}$ may be replaced by $-\boldsymbol{q}$, we may also write

$$W(\boldsymbol{\alpha}; x) = \sum_{\boldsymbol{q}(m)} \cos(2\pi \boldsymbol{\alpha} \cdot \boldsymbol{q})\, {}_pF_{p+1}[(a_p); (b_{p+1}); -x^2q^2]. \tag{1.1b}$$

For conciseness in what follows we define

$$\Delta \equiv \sum_{k=1}^{p+1} b_k - \sum_{k=1}^{p} a_k. \tag{1.2}$$

Ordinary convergence of the series defining $W(\boldsymbol{\alpha}; x)$ will be discussed in section 2, where we shall also show that $W(\boldsymbol{\alpha}; x)$ converges absolutely provided that

$$\mathrm{Re}\,(a_k) > \tfrac{1}{2}m \qquad \mathrm{Re}\,(\Delta) > m + \tfrac{1}{2} \tag{1.3}$$

where Δ is given by equation (1.2) and $1 \leqslant k \leqslant p$. When $p = 0$, the penultimate inequality is superfluous and $\mathrm{Re}\,(b_1) > m + \frac{1}{2}$.

Since specializations of the generalized hypergeometric function ${}_pF_{p+1}$ are proportional to, for example, trigonometric functions, Bessel functions of the first kind, Lommel, Struve, and associated Bessel functions, the m-dimensional Fourier series $W(\boldsymbol{\alpha}; x)$ is of a very general nature. Moreover, lattice sums of the type defined in equations (1.1) appear frequently in the study of finite-size effects in systems undergoing phase transitions. Thus, a knowledge of their analytical behaviour in different domains of the variable x is important for understanding the physical behaviour of the given finite-sized system in various temperature domains. See the work of Allen and Pathria [1, 2] for further details and references.

In [3] we derived closed form representations for $W(\boldsymbol{\alpha}; x)$ in the one- and two-dimensional cases which are given respectively below for $x > 0$ and real numbers α and β:

$$\sum_{\ell\in Z} \mathrm{e}^{2\pi\mathrm{i}\alpha\ell}\, {}_pF_{p+1}[(a_p); (b_{p+1}); -x^2\ell^2] = \frac{\sqrt{\pi}}{x}\frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\frac{\Gamma((a_p)-\frac{1}{2})}{\Gamma((b_{p+1})-\frac{1}{2})}$$
$$\times \sum_{\ell\in Z}^{(\alpha+\ell)^2\leqslant x^2/\pi^2} {}_{p+1}F_p\left[\begin{matrix} \frac{3}{2}-(b_{p+1}); \\ \frac{3}{2}-(a_p); \end{matrix}\ \frac{\pi^2}{x^2}(\alpha+\ell)^2\right]$$
$$+\frac{1}{\sqrt{\pi}}\frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\sum_{k=1}^{p}\frac{\Gamma(\frac{1}{2}-a_k)\Gamma((a_p)^*-a_k)}{\Gamma((b_{p+1})-a_k)}\left(\frac{\pi^2}{x^2}\right)^{a_k}$$
$$\times \sum_{\ell\in Z}^{(\alpha+\ell)^2\leqslant x^2/\pi^2} ((\alpha+\ell)^2)^{a_k-\frac{1}{2}}\, {}_{p+1}F_p\left[\begin{matrix} 1+a_k-(b_{p+1}); \\ \frac{1}{2}+a_k, 1+a_k-(a_p)^*; \end{matrix}\ \frac{\pi^2}{x^2}(\alpha+\ell)^2\right] \tag{1.4}$$

$$\sum_{\ell\in Z}\sum_{n\in Z} \mathrm{e}^{2\pi\mathrm{i}(\alpha\ell+\beta n)}\, {}_pF_{p+1}[(a_p); (b_{p+1}); -x^2(\ell^2+n^2)] = \frac{\pi}{x^2}\frac{\prod_{k=1}^{p+1}(b_k-1)}{\prod_{k=1}^{p}(a_k-1)}$$
$$\times \sum_{\ell\in Z}\sum_{n\in Z}^{(\alpha+\ell)^2+(\beta+n)^2\leqslant x^2/\pi^2} {}_{p+1}F_p\left[\begin{matrix} 2-(b_{p+1}); \\ 2-(a_p); \end{matrix}\ \frac{\pi^2}{x^2}((\alpha+\ell)^2+(\beta+n)^2)\right]$$
$$+\frac{1}{\pi}\frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\sum_{k=1}^{p}\frac{\Gamma(1-a_k)\Gamma((a_p)^*-a_k)}{\Gamma((b_{p+1})-a_k)}\left(\frac{\pi^2}{x^2}\right)^{a_k}$$
$$\times \sum_{\ell\in Z}\sum_{n\in Z}^{(\alpha+\ell)^2+(\beta+n)^2\leqslant x^2/\pi^2} ((\alpha+\ell)^2+(\beta+n)^2)^{a_k-1}$$
$$\times\, {}_{p+1}F_p\left[\begin{matrix} 1+a_k-(b_{p+1}); \\ a_k, 1+a_k-(a_p)*; \end{matrix}\ \frac{\pi^2}{x^2}((\alpha+\ell)^2+(\beta+n)^2)\right] \tag{1.5}$$

where for conciseness $\Gamma((a_p)) \equiv \Gamma(a_1), \ldots, \Gamma(a_p)$ and

$$\Gamma((a_p)^*-a_k) \equiv \Gamma(a_1-a_k), \ldots, \Gamma(a_{k-1}-a_k)\Gamma(a_{k+1}-a_k), \ldots, \Gamma(a_p-a_k)$$

both of which reduce to unity when $p = 0$.

Criteria for absolute convergence of the doubly infinite series in equation (1.4) and the doubly infinite double series in equation (1.5) are determined by setting respectively $m = 1, 2$ in the conditional inequalities (1.3). Several authors have deduced various specializations of equations (1.4) and (1.5) by employing different methods. For further details and references see [3].

Now defining the vector $\boldsymbol{\xi}(m) \equiv \boldsymbol{\alpha}(m) + \boldsymbol{q}(m)$, it is easy to see that the one- and two-dimensional cases given respectively by equations (1.4) and (1.5) may be written in a unified way as

$$W(\boldsymbol{\alpha}; x) = \frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\left\{\left(\frac{\sqrt{\pi}}{x}\right)^m \frac{\Gamma((a_p)-\frac{m}{2})}{\Gamma((b_{p+1})-\frac{m}{2})}\right.$$

$$\times \sum_{\boldsymbol{q}(m)}^{\xi^2 \leqslant x^2/\pi^2} {}_{p+1}F_p\left[\begin{matrix} \frac{m+2}{2} - (b_{p+1}); & \frac{\pi^2\xi^2}{x^2} \\ \frac{m+2}{2} - (a_p); & \end{matrix}\right]$$
$$+ \left(\frac{1}{\sqrt{\pi}}\right)^m \sum_{k=1}^{p} \frac{\Gamma(\frac{m}{2} - a_k)\Gamma((a_p)^* - a_k)}{\Gamma((b_{p+1}) - a_k)} \left(\frac{\pi^2}{x^2}\right)^{a_k}$$
$$\times \sum_{\boldsymbol{q}(m)}^{\xi^2 \leqslant x^2/\pi^2} (\xi^2)^{a_k - \frac{m}{2}} {}_{p+1}F_p\left[\begin{matrix} 1 + a_k - (b_{p+1}); & \frac{\pi^2\xi^2}{x^2} \\ \frac{2-m}{2} + a_k, 1 + a_k - (a_p)*; & \end{matrix}\right]\Bigg\} \tag{1.6}$$

where for absolute convergence of $W(\boldsymbol{\alpha}; x)$ the conditional inequalities (1.3) hold true. Furthermore, by using a representation for the Mellin transform of products of Bessel functions and generalized hypergeometric functions ${}_pF_{p+1}$ (see [4, equations (4.4), (5.1)]), equation (1.6) may be written equivalently in the following elegant and useful form:

$$W(\boldsymbol{\alpha}; x) = \frac{2\pi^{-\frac{m}{2}}}{\Gamma(\frac{m}{2})} \sum_{\boldsymbol{q}(m)}^{\xi^2 \leqslant x^2/\pi^2} \int_0^\infty t^{m-1} {}_0F_1\left[\begin{matrix} -; & -\xi^2 t^2 \\ \frac{m}{2}; & \end{matrix}\right] {}_pF_{p+1}\left[\begin{matrix} (a_p); & -\frac{x^2t^2}{\pi^2} \\ (b_{p+1}); & \end{matrix}\right] \mathrm{d}t \tag{1.7a}$$

where for convergence of the integral when $\xi^2 < x^2/\pi^2$

$$\mathrm{Re}\,(a_k) > \tfrac{1}{4}(m-1) \qquad \mathrm{Re}\,(\Delta) > \tfrac{1}{2}m \tag{1.7b}$$

and when $\xi^2 \leqslant x^2/\pi^2$

$$\mathrm{Re}\,(a_k) > \tfrac{1}{4}(m-1) \qquad \mathrm{Re}\,(\Delta) > \tfrac{1}{2}m + 1 \tag{1.7c}$$

where Δ is given by equation (1.2) and $1 \leqslant k \leqslant p$.

Although in section 4 we shall show by induction that equations (1.7) (and therefore equation (1.6) also) are valid for all dimensions $m \geqslant 1$, we give in section 3 a direct derivation of equation (1.6) for the three-dimensional case. Not only is this derivation interesting in its own right, but it should serve to intimate a direct proof in the m-dimensional case. Assuming therefore that equations (1.6) and (1.7) are valid for m-dimensions, we shall discuss convergence of the lattice sum $W(\boldsymbol{\alpha}; x)$ in the next section.

In equation (1.6) letting $p = 0$, $b_1 = 1 + \nu$, noting that

$${}_0F_1[-; 1+\nu; -z^2] = \Gamma(1+\nu)J_\nu(2z)/z^\nu$$

and using equation (1.1b) we obtain immediately a representation for m-dimensional Schlömilch series:

$$\sum_{\boldsymbol{q}(m)} \cos(2\pi\boldsymbol{\alpha}\cdot\boldsymbol{q}) \frac{J_\nu(2xq)}{(xq)^\nu} = \frac{\pi^{-\frac{m}{2}}}{\Gamma(\frac{2-m}{2}+\nu)} \left(\frac{\pi^2}{x^2}\right)^\nu \sum_{\boldsymbol{q}(m)}^{\xi^2 \leqslant x^2/\pi^2} \left(\frac{x^2}{\pi^2} - \xi^2\right)^{\nu - \frac{m}{2}} \tag{1.8}$$

where (from lemma 1 below) $\mathrm{Re}\,\nu > \frac{m}{2} - 1$ if $\xi^2 < x^2/\pi^2$ or $\mathrm{Re}\,\nu > \frac{m}{2}$ if $\xi^2 \leqslant x^2/\pi^2$. Essentially this is the same result derived previously by induction in [5, equation (2.3)], where the specialized components of the vector $\boldsymbol{\alpha}(m)$ were $\frac{1}{2}$. When $p > 0$, equation (1.6) does not appear suitable for induction because its second right-hand term causes seemingly intractable problems. However, as we shall see in section 4 the equivalent form equation (1.7a) is amenable to the inductive method used previously in [5] to obtain equation (1.8). Furthermore, in [5] we showed how the particular case of equation (1.8) with $\boldsymbol{\alpha}(m) = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ yields a countably infinite number of representations for null-functions on increasingly larger open intervals $0 < x < \pi\alpha$. Thus in section 5 we shall also be able to discuss null-functions in the more general settings of equations (1.6) and (1.7).

2. Convergence of the sum $W(\alpha; x)$

We shall need the asymptotic result (see [4, equation (2.2*a*)]) for the generalized hypergeometric function

$$
{}_pF_{p+1}[(a_p); (b_{p+1}); -z^2] = \left\{ \sum_{k=1}^{p} A_k \left(\frac{1}{z^2}\right)^{a_k} + A_{p+1}\left(\frac{1}{z^2}\right)^{\eta} \times \cos\left[2z - \pi\eta + \mathcal{O}\left(\frac{1}{z}\right)\right]\right\}\left[1 + \mathcal{O}\left(\frac{1}{z^2}\right)\right] \tag{2.1a}
$$

where $|z| \to \infty$, $|\arg z| < \frac{1}{2}\pi$, the A_k $(1 \leqslant k \leqslant p+1)$ are constants dependent on the parameters of the generalized hypergeometric function, and

$$
\eta \equiv \tfrac{1}{2}(\Delta - \tfrac{1}{2}) \tag{2.1b}
$$

where Δ is given by equation (1.2).

Now setting $z = xq(m)$ in equation (2.1*a*), multiplying both sides of the latter by $\exp(2\pi i \boldsymbol{\alpha}(m) \cdot \boldsymbol{q}(m))$, and for some sufficiently large integer $N > 0$ summing the result over $q > N$, we have for $x > 0$

$$
\sum_{q>N} \exp(2\pi i \boldsymbol{\alpha} \cdot \boldsymbol{q})\, {}_pF_{p+1}[(a_p); (b_{p+1}); -q^2x^2]
= \left\{ \sum_{k=1}^{p} \frac{A_k}{x^{2a_k}} \sum_{q>N} \frac{\exp(2\pi i \boldsymbol{\alpha}\cdot\boldsymbol{q})}{(q^2)^{a_k}} + \frac{1}{2}\frac{A_{p+1}}{x^{2\eta}}\left(e^{-i\omega} \sum_{q>N} \frac{\exp[2i(\pi\boldsymbol{\alpha}\cdot\boldsymbol{q} + qx)]}{(q^2)^{\eta}} + e^{i\omega} \sum_{q>N} \frac{\exp[2i(\pi\boldsymbol{\alpha}\cdot\boldsymbol{q} - qx)]}{(q^2)^{\eta}}\right)\right\}\left[1 + \mathcal{O}\left(\frac{1}{x^2}\right)\right] \tag{2.2a}
$$

where $\omega \equiv \pi\eta - \mathcal{O}(1/x)$ and η is given by equation (2.1*b*).

Since $\boldsymbol{\alpha}(m)$ is a constant vector whose m components are arbitrary real numbers, we see without loss of generality that the convergence of $W(\boldsymbol{\alpha}; x)$ is determined by the convergence of the three q-summations in equation (2.2*a*). Clearly the third sum need not be considered separately so for conciseness, we name the first and second q-summations on the right-hand side of equation (2.2*a*) S and T respectively, where in the sum T values of $x \neq 0$ are real. Obviously, necessary conditions that S and T converge respectively are

$$
\mathrm{Re}\,(a_k) > 0\ (1 \leqslant k \leqslant p) \qquad \mathrm{Re}\,(\Delta) > \tfrac{1}{2} \tag{2.2b}
$$

the latter of which follows from equation (2.1*b*), since $\mathrm{Re}\,(\eta) > 0$.

When $m \leqslant 2$, the inequalities $\mathrm{Re}\,(a_k) > 0$ $(1 \leqslant k \leqslant p)$ guarantee the ordinary convergence of S provided that the components of $\boldsymbol{\alpha}(2)$ are not members of Z (see [3, section 2]). Thus, it is reasonable to conjecture that generally in the m-dimensional case, the latter inequalities insure the ordinary convergence of S provided that the components of $\boldsymbol{\alpha}(m)$ are not integers. Furthermore, both S and T converge absolutely respectively provided that

$$
\mathrm{Re}\,(a_k) > \tfrac{1}{2}m\ (1 \leqslant k \leqslant p) \qquad \mathrm{Re}(\Delta) > m + \tfrac{1}{2} \tag{2.2c}
$$

(see [7, p 52]), the latter of which follows from equation (2.1*b*), since $\mathrm{Re}\,(\eta) > \frac{1}{2}m$. Thus, for $m \geqslant 1$ when one of the components of $\boldsymbol{\alpha}(m)$ is an integer, we shall require that $\mathrm{Re}\,(a_k) > \frac{1}{2}m$ $(1 \leqslant k \leqslant p)$ for convergence of the sum S.

Since, even in the two-dimensional case, the ordinary convergence of T (and thus $W(\boldsymbol{\alpha}; x)$ also) is problematic (see [3, section 2] for further details and references), we shall follow the procedure employed in [3] by gleaning additional information (albeit heuristic in nature) from the representation for $W(\boldsymbol{\alpha}; x)$ given by equations (1.7). Thus, we conclude that necessary conditions for the ordinary convergence of $W(\boldsymbol{\alpha}; x)$ are that either of the conditional inequalities (1.7*b*) or (1.7*c*) hold true. For dimensions $m > 1$, the latter conditional inequalities are stronger than the inequalities (2.2*b*). Moreover, when $m > 1$ for absolute convergence of $W(\boldsymbol{\alpha}; x)$, since the inequalities (2.2*c*) are stronger than either of the inequalities (1.7*b*) or (1.7*c*), we shall require the former.

We now summarize in the following conjecture the latter remarks; note that in all cases we have used the strongest applicable inequalities.

Conjectural lemma 1. *For $m \geqslant 1$ and $x > 0$, the sum $W(\boldsymbol{\alpha}(m); x)$ converges under the conditions of each of the following four cases where $1 \leqslant k \leqslant p$ and Δ is given by equation (1.2):*

(i) If none of the components of $\boldsymbol{\alpha}(m)$ is an integer and $\xi^2 < x^2/\pi^2$, then

$$\mathrm{Re}\,(a_k) > \tfrac{1}{4}(m-1) \qquad \mathrm{Re}\,(\Delta) > \tfrac{1}{2}m.$$

(ii) If one of the components of $\boldsymbol{\alpha}(m)$ is an integer and $\xi^2 < x^2/\pi^2$, then

$$\mathrm{Re}\,(a_k) > \tfrac{1}{2}m \qquad \mathrm{Re}\,(\Delta) > \tfrac{1}{2}m.$$

(iii) If none of the components of $\boldsymbol{\alpha}(m)$ is an integer and $\xi^2 \leqslant x^2/\pi^2$, then

$$\mathrm{Re}\,(a_k) > \tfrac{1}{4}(m-1) \qquad \mathrm{Re}\,(\Delta) > \tfrac{1}{2}m+1.$$

(iv) If one of the components of $\boldsymbol{\alpha}(m)$ is an integer and $\xi^2 \leqslant x^2/\pi^2$, then

$$\mathrm{Re}\,(a_k) > \tfrac{1}{2}m \qquad \mathrm{Re}\,(\Delta) > \tfrac{1}{2}m+1.$$

Furthermore, for $m \geqslant 1$ and $x > 0$, the sum $W(\boldsymbol{\alpha}(m); x)$ converges absolutely provided that

$$\mathrm{Re}\,(a_k) > \tfrac{1}{2}m \qquad \mathrm{Re}\,(\Delta) > m+\tfrac{1}{2}.$$

It should be emphasized that although lemma 1 is in part heuristic and based implicitly on the formal method used to obtain $W(\boldsymbol{\alpha}; x)$ in section 4, it is wholly true and provable for $m = 1$ (see [3, lemma 1]); and for $m = 2$ coincides with [3, lemma 2].

3. The three-dimensional case

In the three-dimensional case letting $\boldsymbol{\alpha}(3) = (\alpha, \beta, \gamma)$, $\boldsymbol{q}(3) = (\ell, m, n)$ we write equation (1.1*a*) as

$$W(\boldsymbol{\alpha}(3); x) = \sum_{\ell,m,n\in Z} \mathrm{e}^{2\pi \mathrm{i}\alpha\ell}\mathrm{e}^{2\pi \mathrm{i}\beta m}\mathrm{e}^{2\pi \mathrm{i}\gamma n}\,{}_pF_{p+1}[(a_p); (b_{p+1}); -(\ell^2+m^2+n^2)x^2] \tag{3.1}$$

where $x > 0$. Note that here the second (dummy) summation index m should not be confused with the dimension $m = 3$ of the sum W; and α is just the first component of $\boldsymbol{\alpha}(3)$ and not the length of the vector.

We shall employ a form of the three-dimensional Poisson summation formula to obtain a closed-form representation for $W(\boldsymbol{\alpha}(3); x)$. To this end we shall have to evaluate the three-dimensional Fourier transform $\mathcal{F}$ of the generalized hypergeometric function ${}_pF_{p+1}[-u^2]$, where $\boldsymbol{u} = t(x, y, z)$, $t > 0$:

$$\begin{aligned}\mathcal{F}\{{}_pF_{p+1}[(a_p); (b_{p+1}); -t^2(x^2+y^2+z^2)]\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi x}\mathrm{e}^{\mathrm{i}\eta y}\mathrm{e}^{\mathrm{i}\omega z} \\ &\quad\times {}_pF_{p+1}[(a_p); (b_{p+1}); -t^2(x^2+y^2+z^2)]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \int_0^{\pi}\int_0^{2\pi}\int_0^{\infty} \mathrm{e}^{\mathrm{i}\rho\sin\phi(\xi\cos\theta+\eta\sin\theta)}\mathrm{e}^{\mathrm{i}\rho\omega\cos\phi} \\ &\quad\times {}_pF_{p+1}[(a_p); (b_{p+1}); -t^2\rho^2]\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi\end{aligned} \tag{3.2}$$

which results from the spherical coordinate transformation

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

where $0 \leqslant \phi \leqslant \pi$, $0 \leqslant \theta \leqslant 2\pi$, and $\rho > 0$.

Since

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}\rho \sin\phi(\xi\cos\theta+\eta\sin\theta)}\,\mathrm{d}\theta = 2\pi J_0\left(\rho\sin\phi\sqrt{\xi^2+\eta^2}\right)$$

equation (3.2) reduces to

$$\mathcal{F}\{{}_pF_{p+1}[(a_p);(b_{p+1});-t^2(x^2+y^2+z^2)]\} = 2\pi\int_0^\infty \rho^2\,{}_pF_{p+1}[(a_p);(b_{p+1});-t^2\rho^2] \times \int_0^\pi \sin\phi\cos(\omega\rho\cos\phi)J_0\left(\rho\sin\phi\sqrt{\xi^2+\eta^2}\right)\mathrm{d}\phi\,\mathrm{d}\rho. \tag{3.3}$$

Now making the transformation $x = \sin\phi$ in the latter inner integral, we may use the tabulated result in [6, vol 2, section 2.12.21, equation (6)] or set $\mu = 0$, $a = \frac{1}{2}\omega\rho$, $b = \frac{1}{2}\rho\sqrt{\xi^2+\eta^2}$ in equation (4.1) below to obtain

$$\int_0^\pi \sin\phi\cos(\omega\rho\cos\phi)J_0\left(\rho\sin\phi\sqrt{\xi^2+\eta^2}\right)\mathrm{d}\phi = \frac{2\sin\left(\rho\sqrt{\xi^2+\eta^2+\omega^2}\right)}{\rho\sqrt{\xi^2+\eta^2+\omega^2}}$$

which when combined with the right-hand side of equation (3.3) gives for the three-dimensional Fourier transform $\mathcal{F}$ of ${}_pF_{p+1}[(a_p);(b_{p+1});-t^2(x^2+y^2+z^2)]$ the result

$$\frac{4\pi}{\sqrt{\xi^2+\eta^2+\omega^2}}\int_0^\infty \rho\sin\left(\rho\sqrt{\xi^2+\eta^2+\omega^2}\right){}_pF_{p+1}[(a_p);(b_{p+1});-t^2\rho^2]\,\mathrm{d}\rho.$$

The latter infinite integral is seen to be a specialization of the Mellin transform of products of Bessel functions and generalized hypergeometric functions which has been used previously in connection with equation (1.7*a*). Thus, again employing [4, equations (4.4)] we see that for $t > 0$

$$\mathcal{F}\{{}_pF_{p+1}[(a_p);(b_{p+1});-t^2(x^2+y^2+z^2)]\} = 0 \qquad 4t^2 < \xi^2+\eta^2+\omega^2 \tag{3.4a}$$

and

$$\begin{aligned}\mathcal{F}\{{}_pF_{p+1}[(a_p);(b_{p+1});-t^2(x^2+y^2+z^2)]\} &= \frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\Bigg(\frac{\Gamma((a_p)-\frac{3}{2})}{\Gamma((b_{p+1})-\frac{3}{2})}\frac{\pi^{\frac{3}{2}}}{t^3}\,{}_{p+1}F_p\left[\begin{matrix}\frac{5}{2}-(b_{p+1}); \\ \frac{5}{2}-(a_p);\end{matrix}\;\frac{\xi^2+\eta^2+\omega^2}{4t^2}\right] \\ &\quad + \frac{8\pi^{\frac{3}{2}}}{(\xi^2+\eta^2+\omega^2)^{3/2}}\sum_{k=1}^{p}\frac{\Gamma(\frac{3}{2}-a_k)\Gamma((a_p)^*-a_k)}{\Gamma((b_{p+1})-a_k)}\left(\frac{\xi^2+\eta^2+\omega^2}{4t^2}\right)^{a_k} \\ &\quad \times {}_{p+1}F_p\left[\begin{matrix}1+a_k-(b_{p+1}); \\ a_k-\frac{1}{2}, 1+a_k-(a_p)^*;\end{matrix}\;\frac{\xi^2+\eta^2+\omega^2}{4t^2}\right]\Bigg) \qquad 4t^2 > \xi^2+\eta^2+\omega^2\end{aligned} \tag{3.4b}$$

where for convergence of the Fourier transform

$$\mathrm{Re}\,(a_k) > \tfrac{1}{2} \qquad \mathrm{Re}\,(\Delta) > \tfrac{3}{2} \tag{3.4c}$$

where Δ is given by equation (1.2) and $1 \leqslant k \leqslant p$.

Inversion of the Fourier transform given by equations (3.4*a*) and (3.4*b*) yields the following integral representation for ${}_pF_{p+1}[-t^2(x^2+y^2+z^2)]$:

$$
\begin{aligned}
&{}_pF_{p+1}[(a_p);(b_{p+1});-t^2(x^2+y^2+z^2)]\\
&\quad=\frac{1}{(2\pi)^3}\frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\left(\frac{\Gamma((a_p)-\frac{3}{2})}{\Gamma((b_{p+1})-\frac{3}{2})}\frac{\pi^{\frac{3}{2}}}{t^3}\iiint_{\Omega(t)}\mathrm{e}^{-\mathrm{i}x\xi}\mathrm{e}^{-\mathrm{i}y\eta}\mathrm{e}^{-\mathrm{i}z\omega}\right.\\
&\quad\times{}_{p+1}F_p\left[\begin{matrix}\frac{5}{2}-(b_{p+1}); & \dfrac{\xi^2+\eta^2+\omega^2}{4t^2}\\ \frac{5}{2}-(a_p); & \end{matrix}\right]\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\omega\\
&\quad+8\pi^{\frac{3}{2}}\sum_{k=1}^{p}\frac{\Gamma(\frac{3}{2}-a_k)\Gamma((a_p)^*-a_k)}{\Gamma((b_{p+1})-a_k)}\left(\frac{1}{4t^2}\right)^{a_k}\\
&\quad\times\iiint_{\Omega(t)}\mathrm{e}^{-\mathrm{i}x\xi}\mathrm{e}^{-\mathrm{i}y\eta}\mathrm{e}^{-\mathrm{i}z\omega}(\xi^2+\eta^2+\omega^2)^{a_k-\frac{3}{2}}\\
&\quad\left.\times{}_{p+1}F_p\left[\begin{matrix}1+a_k-(b_{p+1}); & \dfrac{\xi^2+\eta^2+\omega^2}{4t^2}\\ a_k-\frac{1}{2},1+a_k-(a_p)^*; & \end{matrix}\right]\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\omega\right)
\end{aligned}\tag{3.5}
$$

where $t > 0$, the inequalities (3.4*c*) hold true, and the sphere of integration $\Omega(t)$ is given by $\xi^2+\eta^2+\omega^2 \leqslant 4t^2$.

Next, in equation (3.5) replacing the triple x, y, z respectively by ℓ, m, n, setting $t = x$, and inserting the result into equation (3.1) gives for $x > 0$

$$
\begin{aligned}
W(\boldsymbol{\alpha}(3);x)&=\frac{1}{(2\pi)^3}\frac{\pi^{\frac{3}{2}}}{x^3}\frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\frac{\Gamma((a_p)-\frac{3}{2})}{\Gamma((b_{p+1})-\frac{3}{2})}\\
&\quad\times\iiint_{\Omega(x)}\sum_{\ell\in Z}\mathrm{e}^{\mathrm{i}(2\pi\alpha-\xi)\ell}\sum_{m\in Z}\mathrm{e}^{\mathrm{i}(2\pi\beta-\eta)m}\sum_{n\in Z}\mathrm{e}^{\mathrm{i}(2\pi\gamma-\omega)n}\\
&\quad\times{}_{p+1}F_p\left[\begin{matrix}\frac{5}{2}-(b_{p+1}); & \dfrac{\xi^2+\eta^2+\omega^2}{4x^2}\\ \frac{5}{2}-(a_p); & \end{matrix}\right]\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\omega\\
&\quad+\frac{8\pi^{\frac{3}{2}}}{(2\pi)^3}\frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\sum_{k=1}^{p}\frac{\Gamma(\frac{3}{2}-a_k)\Gamma((a_p)^*-a_k)}{\Gamma((b_{p+1})-a_k)}\left(\frac{1}{4x^2}\right)^{a_k}\\
&\quad\times\iiint_{\Omega(x)}\sum_{\ell\in Z}\mathrm{e}^{\mathrm{i}(2\pi\alpha-\xi)\ell}\sum_{m\in Z}\mathrm{e}^{\mathrm{i}(2\pi\beta-\eta)m}\sum_{n\in Z}\mathrm{e}^{\mathrm{i}(2\pi\gamma-\omega)n}(\xi^2+\eta^2+\omega^2)^{a_k-\frac{3}{2}}\\
&\quad\times{}_{p+1}F_p\left[\begin{matrix}1+a_k-(b_{p+1}); & \dfrac{\xi^2+\eta^2+\omega^2}{4x^2}\\ a_k-\frac{1}{2},1+a_k-(a_p)^*; & \end{matrix}\right]\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\omega
\end{aligned}\tag{3.6}
$$

where the order of summations and integrations have been interchanged in both terms.

Now for real μ noting that

$$\sum_{k\in Z}\mathrm{e}^{\mathrm{i}\mu k}=2\pi\sum_{k\in Z}\delta(\mu-2\pi k)$$

(see [8, p 189, equation (17)]), where δ is the delta function (or functional), upon replacing μ respectively by $2\pi\alpha-\xi$, $2\pi\beta-\eta$, $2\pi\gamma-\omega$ we see that equation (3.6) yields for $x > 0$

$$
\begin{aligned}
W(\boldsymbol{\alpha}(3);x)&=\frac{\pi^{\frac{3}{2}}}{x^3}\frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\frac{\Gamma((a_p)-\frac{3}{2})}{\Gamma((b_{p+1})-\frac{3}{2})}\\
&\quad\times\sum_{\ell,m,n\in Z}\iiint_{\Omega(x)}{}_{p+1}F_p\left[\begin{matrix}\frac{5}{2}-(b_{p+1}); & \dfrac{\xi^2+\eta^2+\omega^2}{4x^2}\\ \frac{5}{2}-(a_p); & \end{matrix}\right]\\
&\quad\times\delta(2\pi\alpha-\xi-2\pi\ell)\delta(2\pi\beta-\eta-2\pi m)\delta(2\pi\gamma-\omega-2\pi n)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\omega
\end{aligned}
$$

$$+8\pi^{\frac{3}{2}}\frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\sum_{k=1}^{p}\frac{\Gamma(\frac{3}{2}-a_k)\Gamma((a_p)^*-a_k)}{\Gamma((b_{p+1})-a_k)}\left(\frac{1}{4x^2}\right)^{a_k}$$
$$\times\sum_{\ell,m,n\in Z}\iiint_{\Omega(x)}(\xi^2+\eta^2+\omega^2)^{a_k-\frac{3}{2}}$$
$$\times{}_{p+1}F_p\left[\begin{matrix}1+a_k-(b_{p+1}); & \frac{\xi^2+\eta^2+\omega^2}{4x^2}\\ a_k-\frac{1}{2},1+a_k-(a_p)^*; & \end{matrix}\right]$$
$$\times\delta(2\pi\alpha-\xi-2\pi\ell)\delta(2\pi\beta-\eta-2\pi m)\delta(2\pi\gamma-\omega-2\pi n)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\omega \qquad (3.7)$$

where again we have interchanged the order of summations and integrations in both terms.

Finally, on performing the required formal term-by-term integrations with regard to the distributional properties of the delta function we deduce for $x > 0$ and real numbers α, β, γ

$$\sum_{\ell,m,n\in Z}\mathrm{e}^{2\pi\mathrm{i}(\alpha\ell+\beta m+\gamma n)}{}_pF_{p+1}[(a_p);(b_{p+1});-x^2(\ell^2+m^2+n^2)]$$
$$=\frac{\pi^{\frac{3}{2}}}{x^3}\frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\frac{\Gamma((a_p)-\frac{3}{2})}{\Gamma((b_{p+1})-\frac{3}{2})}\sum_{\ell,m,n\in Z}^{(\alpha+\ell)^2+(\beta+m)^2+(\gamma+n)^2\leqslant x^2/\pi^2}$$
$${}_{p+1}F_p\left[\begin{matrix}\frac{5}{2}-(b_{p+1}); & \frac{\pi^2}{x^2}((\alpha+\ell)^2+(\beta+m)^2+(\gamma+n)^2)\\ \frac{5}{2}-(a_p); & \end{matrix}\right]$$
$$+\frac{1}{\pi^{\frac{3}{2}}}\frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\sum_{k=1}^{p}\frac{\Gamma(\frac{3}{2}-a_k)\Gamma((a_p)^*-a_k)}{\Gamma((b_{p+1})-a_k)}\left(\frac{\pi^2}{x^2}\right)^{a_k}$$
$$\times\sum_{\ell,m,n\in Z}^{(\alpha+\ell)^2+(\beta+m)^2+(\gamma+n)^2\leqslant x^2/\pi^2}((\alpha+\ell)^2+(\beta+m)^2+(\gamma+n)^2)^{a_k-\frac{3}{2}}$$
$$\times{}_{p+1}F_p\left[\begin{matrix}1+a_k-(b_{p+1}); & \frac{\pi^2}{x^2}((\alpha+\ell)^2+(\beta+m)^2+(\gamma+n)^2)\\ a_k-\frac{1}{2},1+a_k-(a_p)^*; & \end{matrix}\right] \qquad (3.8)$$

where convergence may be determined from lemma 1 for dimension $m = 3$. Thus, it is easy to see that we have verified equation (1.6) in the three-dimensional case where $\xi^2(3) = (\alpha+\ell)^2 + (\beta+m)^2 + (\gamma+n)^2$. In the next section we shall show by induction that equations (1.6) and (1.7) are in fact valid for all dimensions $m \geqslant 1$.

4. The inductive formal proof

In what follows we shall need to utilize

$$\int_0^{\pi/2}\sin^{2\mu+1}\phi\,{}_0F_1\left[\begin{matrix}—; & -a^2\cos^2\phi\\ \frac{1}{2}; & \end{matrix}\right]{}_0F_1\left[\begin{matrix}—; & -b^2\sin^2\phi\\ 1+\mu; & \end{matrix}\right]\mathrm{d}\phi$$
$$=\frac{1}{2}\sqrt{\pi}\frac{\Gamma(1+\mu)}{\Gamma(\frac{3}{2}+\mu)}{}_0F_1\left[\begin{matrix}—; & -(a^2+b^2)\\ \frac{3}{2}+\mu; & \end{matrix}\right] \qquad (4.1)$$

where $\mathrm{Re}\,\mu > -1$. This well known result (cf [6, vol 2, section 2.12.21, equation (5)]) is easily derived by writing each ${}_0F_1$ as a hypergeometric sum, noting that the term-by-term integrations are proportional to beta functions, and then essentially employing the binomial theorem.

We shall also employ the addition theorem for generalized hypergeometric functions (see e.g. [6, vol 3, section 6.8.1, equation (19)])

$${}_pF_q[(a_p);(b_q);x+y]=\sum_{n=0}^{\infty}\frac{((a_p))_n}{((b_q))_n}\frac{x^n}{n!}{}_pF_q[(a_p)+n;(b_q)+n;y] \qquad (4.2)$$

(which essentially is a consequence of the binomial theorem) where for conciseness

$$((a_p))_n \equiv (a_1)_n(a_2)_n, \ldots, (a_p)_n$$

which reduces to unity when $p = 0$.

It is our intention to show that equation (1.7*a*) holds true for all dimensions $m \geqslant 1$. Certainly, it holds true for $m = 1$ as we have stated in section 1. Therefore, assuming equation (1.7*a*) is valid for an arbitrary integer m, if we can show that it is valid for $m + 1$, then by the principle of mathematical induction it is true for all positive integers m. To this end recall that $W(\boldsymbol{\alpha}(m); x)$ is defined by equation (1.1*a*):

$$W(\boldsymbol{\alpha}(m); x) \equiv \sum_{\boldsymbol{q}(m)} \exp(2\pi \mathrm{i}\boldsymbol{\alpha}(m) \cdot \boldsymbol{q}(m))\, {}_pF_{p+1}[(a_p); (b_{p+1}); -x^2q^2(m)].$$

For conciseness letting $S(m) = W(\boldsymbol{\alpha}(m); x)$ we then have

$$S(m+1) = \sum_{\ell \in Z} \mathrm{e}^{2\pi \mathrm{i}\alpha_\ell \ell} \sum_{\boldsymbol{q}(m)} \exp(2\pi \mathrm{i}\boldsymbol{\alpha} \cdot \boldsymbol{q})\, {}_pF_{p+1}[(a_p); (b_{p+1}); -x^2q^2 - x^2\ell^2] \tag{4.3}$$

where α_ℓ is an arbitrary ℓth component of the vector $\boldsymbol{\alpha}(m+1)$, the index ℓ is the ℓth component of $\boldsymbol{q}(m+1)$, and on the right-hand side of equation (4.3) it is understood that $\boldsymbol{\alpha} = \boldsymbol{\alpha}(m)$, $\boldsymbol{q} = \boldsymbol{q}(m)$. Now employing the addition theorem for generalized hypergeometric functions given by equation (4.2), we see that equation (4.3) yields

$$S(m+1) = \sum_{\ell \in Z} \mathrm{e}^{2\pi \mathrm{i}\alpha_\ell \ell} \sum_{n=0}^{\infty} \frac{((a_p))_n}{((b_{p+1}))_n} \frac{(-x^2\ell^2)^n}{n!} \times \sum_{\boldsymbol{q}(m)} \exp(2\pi \mathrm{i}\boldsymbol{\alpha} \cdot \boldsymbol{q})\, {}_pF_{p+1}\left[\begin{matrix}(a_p)+n; \\ (b_{p+1})+n;\end{matrix}\; -x^2q^2\right] \tag{4.4}$$

where the order of the second and third summations have been interchanged.

Since obviously the $\boldsymbol{q}(m)$-summation in equation (4.4) is m-dimensional, it may be evaluated by using the induction hypothesis given by equation (1.7*a*) with (a_p) replaced by $(a_p) + n$ and (b_{p+1}) replaced by $(b_{p+1}) + n$. Thus we have

$$S(m+1) = \frac{2\pi^{-\frac{m}{2}}}{\Gamma(\frac{m}{2})} \sum_{\ell \in Z} \mathrm{e}^{2\pi \mathrm{i}\alpha_\ell \ell} \sum_{n=0}^{\infty} \frac{((a_p))_n}{((b_{p+1}))_n} \frac{(-x^2\ell^2)^n}{n!} \times \sum_{\boldsymbol{q}(m)}^{\xi^2 \leqslant x^2/\pi^2} \int_0^\infty t^{m-1}\, {}_0F_1\left[\begin{matrix}-; \\ \frac{m}{2};\end{matrix}\; -\xi^2t^2\right] {}_pF_{p+1}\left[\begin{matrix}(a_p)+n; \\ (b_{p+1})+n;\end{matrix}\; \frac{-x^2t^2}{\pi^2}\right] \mathrm{d}t \tag{4.5}$$

where $\xi(m) = |\boldsymbol{\alpha}(m) + \boldsymbol{q}(m)|$. Now using the elementary identity

$$(\alpha + n)_k = (\alpha)_k(\alpha + k)_n/(\alpha)_n$$

the generalized hypergeometric function ${}_pF_{p+1}[-x^2t^2/\pi^2]$ may be written as

$${}_pF_{p+1}\left[\begin{matrix}(a_p)+n; \\ (b_{p+1})+n;\end{matrix}\; \frac{-x^2t^2}{\pi^2}\right] = \frac{((b_{p+1}))_n}{((a_p))_n} \sum_{k=0}^{\infty} \frac{((a_p))_k}{((b_{p+1}))_k} \frac{((a_p)+k)_n}{((b_{p+1})+k)_n} \frac{(-x^2t^2/\pi^2)^k}{k!}$$

which when inserted into equation (4.5) gives

$$S(m+1) = \frac{2\pi^{-\frac{m}{2}}}{\Gamma(\frac{m}{2})} \sum_{\boldsymbol{q}(m)}^{\xi^2 \leqslant x^2/\pi^2} \int_0^\infty t^{m-1}\, {}_0F_1\left[-; \frac{m}{2}; -\xi^2t^2\right] \sum_{k=0}^{\infty} \frac{((a_p))_k}{((b_{p+1}))_k} \frac{(-x^2t^2/\pi^2)^k}{k!} \times \sum_{\ell \in Z} \mathrm{e}^{2\pi \mathrm{i}\alpha_\ell \ell} \sum_{n=0}^{\infty} \frac{((a_p)+k)_n}{((b_{p+1})+k)_n} \frac{(-x^2\ell^2)^n}{n!}\, \mathrm{d}t \tag{4.6}$$

where, in addition to interchanges in the order of integration and ℓ, n-summations, there have occurred interchanges in the orders of the summations themselves.

Next, observing that the n-summation in equation (4.6) is just

$$ {}_pF_{p+1}[(a_p)+k;\,(b_{p+1})+k;\,-x^2\ell^2] $$

we may evaluate the one-dimensional ℓ-summation by again using equation (1.7a) now with $m=1$, $\alpha=\alpha_\ell$, (a_p) replaced by $(a_p)+k$, and (b_{p+1}) replaced by $(b_{p+1})+k$ thus giving

$$ \sum_{\ell\in Z} \mathrm{e}^{2\pi\mathrm{i}\alpha_\ell\ell}\,{}_pF_{p+1}[(a_p)+k;\,(b_{p+1})+k;\,-x^2\ell^2] $$
$$ = \frac{2}{\pi}\sum_{\ell\in Z}^{(\alpha_\ell+\ell)^2\leqslant x^2/\pi^2}\int_0^\infty {}_0F_1\left[\begin{matrix}-;\\ \frac{1}{2};\end{matrix}\;-(\alpha_\ell+\ell)^2s^2\right]{}_pF_{p+1} \times\left[\begin{matrix}(a_p)+k;\\ (b_{p+1})+k;\end{matrix}\;\frac{-x^2s^2}{\pi^2}\right]\mathrm{d}s \qquad (4.7) $$

where on the right-hand side we have renamed the one component vector $\boldsymbol{q}(1)$ by again using the scaler index ℓ. Thus equations (4.6) and (4.7) yield

$$ S(m+1) = \frac{4\pi^{-\frac{m}{2}-1}}{\Gamma(\frac{m}{2})}\sum_{\boldsymbol{q}(m)}^{\xi^2\leqslant x^2/\pi^2}\sum_{\ell\in Z}^{(\alpha_\ell+\ell)^2\leqslant x^2/\pi^2}\int_0^\infty t^{m-1}{}_0F_1\left[-;\frac{m}{2};-\xi^2t^2\right] \times\int_0^\infty {}_0F_1\left[-;\frac{1}{2};-(\alpha_\ell+\ell)^2s^2\right]\sum_{k=0}^\infty\frac{((a_p))_k}{((b_{p+1}))_k}\frac{(-x^2t^2/\pi^2)^k}{k!} \times{}_pF_{p+1}\left[\begin{matrix}(a_p)+k;\\ (b_{p+1})+k;\end{matrix}\;\frac{-x^2s^2}{\pi^2}\right]\mathrm{d}s\,\mathrm{d}t \qquad (4.8) $$

where there have occurred changes in the order of s-integration and k-summation along with changes in the order of t-integration and ℓ-summation.

The k-summation in equation (4.8) is simply ${}_pF_{p+1}[(a_p);(b_{p+1});\frac{-x^2}{\pi^2}(s^2+t^2)]$ (cf equation (4.2)) so that the latter result may be written as

$$ S(m+1) = \frac{4\pi^{-\frac{m}{2}-1}}{\Gamma(\frac{m}{2})}\sum_{\boldsymbol{q}(m)}^{\xi^2\leqslant x^2/\pi^2}\sum_{\ell\in Z}^{(\alpha_\ell+\ell)^2\leqslant x^2/\pi^2}\int_0^\infty\int_0^\infty t^{m-1}{}_0F_1\left[-;\frac{m}{2};-\xi^2t^2\right] \times{}_0F_1[-;\tfrac{1}{2};-(\alpha_\ell+\ell)^2s^2]{}_pF_{p+1}\left[(a_p);(b_{p+1});\frac{-x^2}{\pi^2}(s^2+t^2)\right]\mathrm{d}s\,\mathrm{d}t. \qquad (4.9) $$

Now making the polar coordinate transformation $s=r\cos\phi$, $t=r\sin\phi$, the double integral in equation (4.9) becomes

$$ \int_0^\infty r^m{}_pF_{p+1}[(a_p);(b_{p+1});-x^2r^2/\pi^2]\int_0^{\pi/2}\sin^{m-1}\phi \times{}_0F_1\left[-;\frac{m}{2};-\xi^2r^2\sin^2\phi\right]{}_0F_1\left[-;\frac{1}{2};-(\alpha_\ell+\ell)^2r^2\cos^2\phi\right]\mathrm{d}\phi\,\mathrm{d}r. $$

The latter inner integral is evaluated by using equation (4.1) with $\mu=\frac{1}{2}(m-2)$, $a=(\alpha_\ell+\ell)r$, $b=\xi r$ thus giving

$$ \frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m+1}{2})}\int_0^\infty r^m{}_0F_1\left[-;\frac{m+1}{2};-(\xi^2+(\alpha_\ell+\ell)^2)r^2\right]{}_pF_{p+1}\left[(a_p);(b_{p+1});\frac{-x^2r^2}{\pi^2}\right]\mathrm{d}r $$

where $\xi=\xi(m)$.

Since $S(m+1) = W(\boldsymbol{\alpha}(m+1); x)$, letting $\xi^2(m+1) = \xi^2(m) + (\alpha_\ell + \ell)^2$ we may then write equation (4.9) as

$$W(\boldsymbol{\alpha}(m+1); x) = \frac{2\pi^{-\frac{m+1}{2}}}{\Gamma(\frac{m+1}{2})} \sum_{\boldsymbol{q}(m+1)}^{\xi^2 \leqslant x^2/\pi^2} \int_0^\infty r^m {}_0F_1\left[\begin{matrix} —; \\ \frac{m+1}{2}; \end{matrix} \; -\xi^2 r^2\right] {}_pF_{p+1}\left[\begin{matrix} (a_p); \\ (b_{p+1}); \end{matrix} \; \frac{-x^2 r^2}{\pi^2}\right] \mathrm{d}r$$

where $\xi = \xi(m+1)$. Using once again [4, equations (4.4), (5.1)] the latter integral converges when $\xi^2 < x^2/\pi^2$ provided that

$$\mathrm{Re}\,(a_k) > \tfrac{1}{4}m \qquad \mathrm{Re}\,(\Delta) > \tfrac{1}{2}(m+1)$$

and when $\xi^2 \leqslant x^2/\pi^2$ provided that

$$\mathrm{Re}\,(a_k) > \tfrac{1}{4}m \qquad \mathrm{Re}\,(\Delta) > \tfrac{1}{2}(m+3)$$

where Δ is given by equation (1.2) and $1 \leqslant k \leqslant p$. Thus we have reproduced above equations (1.7) with m replaced by $m+1$. This evidently completes the inductive proof of equations (1.7).

5. Null-functions

Recall that the vector $\boldsymbol{\xi}(m)$ is defined by

$$\boldsymbol{\xi}(m) = \boldsymbol{\alpha}(m) + \boldsymbol{q}(m)$$

where the m components of $\boldsymbol{\alpha}(m)$ are arbitrary real numbers and the m components of $\boldsymbol{q}(m)$ are integers in Z. If we let r be a positive integer greater than one, $q_i \in Z$ $(1 \leqslant i \leqslant m)$, and set

$$\begin{aligned} \boldsymbol{\alpha}(m) &= \left(\frac{1}{r}, \frac{1}{r}, \ldots, \frac{1}{r}\right) \\ \boldsymbol{q}(m) &= (q_1, q_2, \ldots, q_m) \end{aligned} \tag{5.1}$$

then

$$\xi^2 = \frac{1}{r^2}[(1+rq_1)^2 + (1+rq_2)^2 + \cdots + (1+rq_m)^2]$$

where $\xi = \xi(m)$ is the length of $\boldsymbol{\xi}(m)$.

Now if x is the interval $0 < x < \pi\sqrt{m}/r$, such that $\xi^2 < x^2/\pi^2$, then we have

$$(1+rq_1)^2 + (1+rq_2)^2 + \cdots + (1+rq_m)^2 < m. \tag{5.2}$$

Furthermore, since none of the components of $\boldsymbol{\alpha}(m)$ given by equation (5.1) is an integer and $\xi^2 < x^2/\pi^2$, from lemma 1(i) the sum $W(\boldsymbol{\alpha}; x)$ exists provided that for $1 \leqslant k \leqslant p$

$$\mathrm{Re}\,(a_k) > \tfrac{1}{4}(m-1) \qquad \mathrm{Re}\,(\Delta) > \tfrac{1}{2}m \tag{5.3}$$

where Δ is given by equation (1.2). However, because the finite sums on the right-hand sides of either equations (1.6) or (1.7*a*) are empty (since the inequality (5.2) can never be satisfied), it is evident that $W(\boldsymbol{\alpha}; x) = 0$. Thus observing that $\alpha^2 = m/r^2$ we have the following corollary.

Corollary 1. *Suppose for positive integer r greater than one that $\boldsymbol{\alpha}(m)$ is given by equation (5.1). Then*

$$\sum_{\boldsymbol{q}(m)} \cos(2\pi\boldsymbol{\alpha}\cdot\boldsymbol{q})\, {}_pF_{p+1}[(a_p); (b_{p+1}); -x^2q^2] = 0 \tag{5.4}$$

for all x is the open interval $(0, \pi\alpha)$ provided that the conditional inequalities (5.3) hold true.

Thus for fixed integer $r > 1$ equation (5.4) provides a countably infinite number of representations for null-functions on increasingly larger open intervals. Perhaps more remarkable is the fact that these null-functions $W(\boldsymbol{\alpha}; x)$ are parametrically independent of the choice of generalized hypergeometric functions generating them.

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